Hypothesis Testing

ST551 Lecture 8

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2017-10-06

Inference: Hypothesis Tests

Key Components

- Hypotheses
  - Null Hypothesis
  - Alternative Hypothesis
- Test Statistic
- Reference Distribution (Null Distribution)
  - Rejection Region

Possible outcomes

	H_0 is true	H_A is true
Reject Null	Type I error	Correct decision
Fail to reject Null	Correct decision	Type II error

Balancing Type I and Type II error

Hypothesis tests are designed to control Type I error

The **significance level**, α is the probability of a Type I error:

$$\alpha = P\,(\text{Reject } H_0 \text{ when } H_0 \text{ is true}) = P_{H_0}(\text{Reject } H_0)$$

Common choices: $\alpha = 0.05$, $\alpha = 0.01$

Power

The **power**, of a test is probability of *correctly* rejecting the null hypothesis (H_0), but is a function of some actual parameter value (θ_1).

$$\begin{aligned} \text{Power}(\theta_1) &= P\left(\text{Reject } H_0 \text{ when } \theta_1 \text{ is true}\right) \\ &= P_{\theta_1}(\text{Reject } H_0) \\ &= 1 - \beta(\theta_1) \end{aligned}$$

for $\theta_1 \in H_A$ where $\beta(\theta_1)$ = Probability of Type II error

Statistic

Test statistic: A statistic $T(Y_1, \ldots, Y_n)$ (i.e. a function of the sample values) that is used to make the decision on whether or not to reject the null hypothesis.

We want the test statistic to capture the evidence the data provides about the hypotheses.

Null Distribution/Rejection Region

Reference Distribution (Null distribution) The distribution the test statistic will be compared to.

Usually, the distribution of the test statistic when the null hypothesis is true.

Rejection region: values of the test statistic that will result in **rejecting the null** hypothesis.

How do we find it? Consider values of the test statistic which are *most* unusual and would be more typical if the alternative were true.

Example: rejection region calculation

Ex: Would you reject $H_0 : \mu = 30$ in favor of $H_A : \mu < 30$ if your sample of 100 OSU freshman had a sample mean hours spent preparing for class:

$\overline{Y}$=31?

$\overline{Y}$=25?

$\overline{Y}$=29.2?

To decide we need to use the **sampling distribution of the sample mean**.

(We will assume, for now, that the population variance $\sigma^2 = 25$ *is known)*

Critical value

The **rejection region** is of one of these forms:

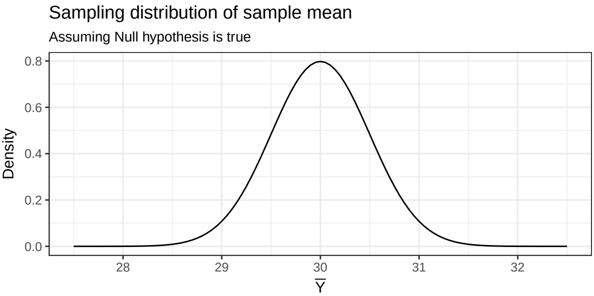
- Reject H_0 if $T > c_U$
- Reject H_0 if $T < c_L$
- Reject H_0 if $T > c_U$ or $T < c_L$

c_U c_L are called **critical values** for the test and are chosen to obtain the desired significance level (i.e. to control the Type I error rate).

Example: continued

Consider values of the test statistic which are most unusual and would be more typical if the alternative were true.

If null hypothesis were true: $\overline{Y} \dot{\sim} N(\quad , \qquad)$



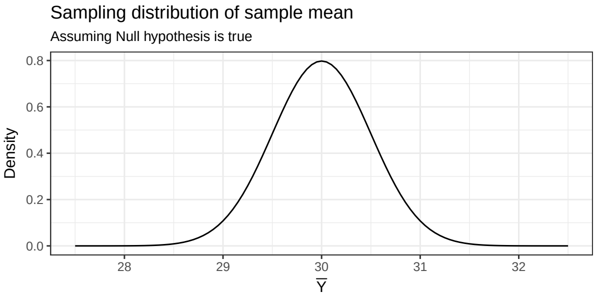
More typical values under the alternative would be on the low side of the distribution.

Example: continued

Rejection region will be of the form: Reject H_0 if $T < c_L$

> *c_U c_L are are chosen to obtain the desired significance level.*

What value of c_L, gives $P_{H_0}(\text{Reject } H_0) = \alpha = 0.05$?



`qnorm(0.05, mean = 30, sd = sqrt(25/100))` $= 29.18$

Example: continued

Reject H_0 if $\overline{Y} < 29.18$

Ex: Would you reject $H_0 : \mu = 30$ in favor of $H_A : \mu < 30$ if your sample of 100 OSU freshman had a sample mean hours spent preparing for class:

$\overline{Y}$=31? **No**

$\overline{Y}$=25? **Yes**

$\overline{Y}$=29.2? **No**

Z-test

If we wanted to use a *Standard Normal* as the reference distribution, we would standardize the sample mean by its mean and standard deviation under the null.

I.e. subtract the hypothesized mean μ_0 and divide by $\sqrt{\sigma^2/n}$

$$Z(\mu_0) = \frac{\overline{Y} - \mu_0}{\sqrt{\sigma^2/n}}$$

Leads to the **Z-test**.

To test a hypothesis about population mean when population variance is known.

To test $H_0 : \mu = \mu_0$

1. Find test statistic:

$$Z(\mu_0) = \frac{\overline{Y} - \mu_0}{\sqrt{\sigma^2/n}}$$

2. Compare to Standard Normal

Z-test: Critical values

- $H_A : \mu > \mu_0$, reject H_0 when $Z(\mu_0) > z_{1-\alpha}$
- $H_A : \mu < \mu_0$, reject H_0 when $Z(\mu_0) < z_{\alpha}$
- $H_A : \mu \neq \mu_0$ reject H_0 when $Z(\mu_0) < z_{\alpha/2}$ or $Z(\mu_0) > z_{1-\alpha/2}$, equivalently $|Z(\mu_0)| > z_{1-\alpha/2}$

z_{α}, is the value z such that $P(Z < z) = \alpha$ where $Z \sim N(0,1)$, can find with in R `qnorm(alpha)`.

Z-test: Recap

Data Setting One sample, no explanatory variable $Y_1, \ldots, Y_n$ i.i.d from population with known variance σ^2

Null hypothesis $H_0 : \mu = \mu_0$

Test statistic

$$Z(\mu_0) = \frac{\overline{Y} - \mu_0}{\sqrt{\sigma^2/n}}$$

Reference distribution $Z(\mu_0) \dot{\sim} N(0, 1)$

Rejection Region

One sided $H_A : \mu < \mu_0$	Two sided $H_A : \mu \neq \mu_0$	One sided $H_A : \mu > \mu_0$
$Z(\mu_0) < z_\alpha$	$\lvert Z(\mu_0)\rvert > z_{1-\alpha/2}$	$Z(\mu_0) > z_{1-\alpha}$

How do we know if a hypothesis test is good?

Exactness Is the actual rejection probability equal to the significance level α?

- **Finite sample exactness**: For finite samples of size n is $P(\text{Reject } H_0) = \alpha$ when null is true?
- **Asymptotic exactness**: As n goes to infinity does $P(\text{Reject } H_0) \to \alpha$ when null is true?

A test is finite-sample exact if reference distribution is exactly the sampling distribution for test statistic when null is true.

A test is asymptotically exact if reference distribution is the **asymptotic** sampling distribution for test statistic when null is true.

Your turn

Is the Z-test:

- finite sample exact?
  When is the sampling distribution of $\overline{Y}$ exactly $N(\mu_0, \sigma^2/n)$?
  Only when population distribution is Normal, i.e
  $Y \sim N(\mu_0, \sigma^2)$
- asymptotic exact?
  When does the sampling distribution of $\overline{Y}$ approach
  $N(\mu_0, \sigma^2/n)$ as $n \to \infty$?
  ALWAYS! Thanks to the CLT.

How do we know if a hypothesis test is good?

Consistency

For any fixed setting where alternative is true, does the rejection probability tend to one as sample approaches infinity?

$$\text{Power}(\theta_1) \to_p 1, \quad \text{for any } \theta_1 \in H_A$$

(If we can take an infinite sample are we guaranteed to reject null when alternative is in fact true.)

Is the Z-test consistent?

What is the power of the test? $\mu = \mu_A \neq \mu_0$

Depends on rejection region (let's do one sided upper):

$$
\begin{aligned}
P\left(Z(\mu_0) > z_{1-\alpha}\right) &= P\left(\frac{\overline{Y} - \mu_0}{\sqrt{\sigma^2/n}} > z_{1-\alpha}\right) \\
&= P\left(\overline{Y} > z_{1-\alpha}\sqrt{\sigma^2/n} + \mu_0\right) \\
&= P\left(\frac{\overline{Y} - \mu_A}{\sqrt{\sigma^2/n}} > z_{1-\alpha} + \frac{\mu_0 - \mu_A}{\sqrt{\sigma^2/n}}\right) \\
&= P\left(\frac{\overline{Y} - \mu_A}{\sqrt{\sigma^2/n}} > z_{1-\alpha} - \frac{\sqrt{n}(\mu_A - \mu_0)}{\sqrt{\sigma^2}}\right)
\end{aligned}
$$

Is the Z-test consistent?

Upper alternative $\mu_A - \mu_0 > 0$, as $n \to \infty$ subtracting a bigger and bigger term from critical value. So, term on right gets smaller as n gets larger.

$$1 - \Phi\left(z_{1-\alpha} - \frac{\sqrt{n}(\mu_A - \mu_0)}{\sqrt{\sigma^2}}\right)$$

$$\to 1 - \Phi(-\infty) = 1 - 0 = 1$$

Yes, Z-test is consistent