T-tests

ST551 Lecture 10

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Warm up: from last times slides:

A random sample of $n = 25$ Corvallis residents had an average *IQ score* of 104. Assume a population variance of $\sigma^2 = 225$. What's the mean IQ for Corvallis residents? Is it plausible the mean for Corvallis residents is greater than 100?

Find point estimate, Z-stat and p-value, and 95% confidence interval

`qnorm(0.975)` $= 1.96$

`pnorm(1.33)` $= 0.9082409$

Finish last time's slides

t-tests

Inference for a population mean

What do we do if we don't know σ^2? *Realistically this is always the case*

We can estimate σ^2, just like we estimated μ:

- We used the sample mean to estimate the population mean
- We can use the sample variance to estimate the population variance

Sample variance

The sample variance for a sample $Y_1, \ldots, Y_n$ is:

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(Y_i - \overline{Y} \right)^2$$

Facts about the sampling distribution of the sample variance:

- The mean is σ^2, i.e. s^2 is an unbiased estimate of σ^2
- As the sample size n gets larger, s^2 gets closer and closer to the true population variance σ^2, i.e. s^2 is consistent estimate of σ^2

t-statistic

If we replace σ^2 with s^2 in the Z-statistic for testing $H_0 : \mu = \mu_0$, we get a t-statistic:

$$Z(\mu_0) = \frac{\overline{Y} - \mu_0}{\sqrt{\sigma^2/n}} \rightarrow t(\mu_0) = \frac{\overline{Y} - \mu_0}{\sqrt{s^2/n}}$$

Reference distribution

We compared the Z-statistic to $\mathrm{N}(0, 1)$

- **Why?** $\mathrm{N}(0, 1)$ is the distribution we expect for Z, when the null hypothesis is true

What should we compare a t-statistic to?

- s^2 will sometimes be smaller than σ^2, sometimes bigger
- Introduces additional variability into our test statistic

t-distribution

The null distribution for a t-statistic is the t-distribution.

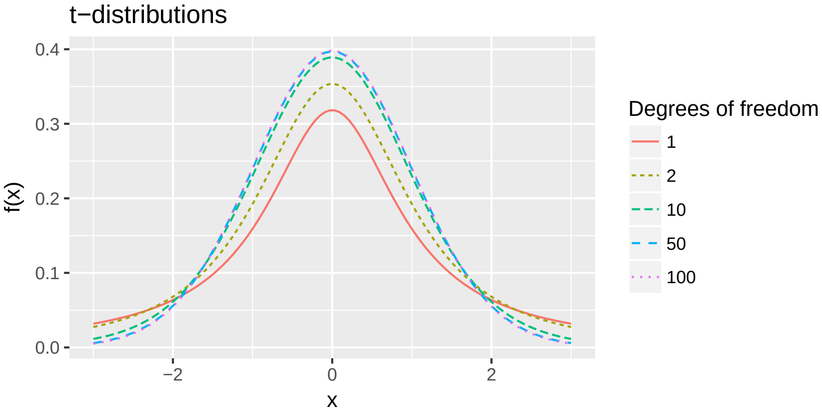
the t-distribution is a family of distributions defined by a single parameter the *degrees of freedom*

Notation:

- $t_($1) - t-distribution with 1 degree of freedom
- $t_($3) - t-distribution with 3 degrees of freedom
- $t_(v)$ - t-distribution with v degree of freedom

Looks a lot like a Standard normal but with heavier tails and sharper peak.

t-distribution



As $v \to \infty$ $t_{(v)}$ approaches the Standard Normal density.

t-test: Inference for population mean

- If the population is exactly Normal:
  - $\overline{Y}$ exactly Normal.
  - t-statistic is exactly a t-distribution with $n - 1$ degrees of freedom
- If population is anything with finite variance:
  - $\overline{Y}$ approximately Normal,
  - t-statistic approximately t-distribution with $n - 1$ d.f.

t-test: Inference for population mean

Rather than coming from a Standard Normal:

- Rejection region critical values come from t-distribution quantiles
- CI multipliers come from t-distribution quantiles
- P-values come from the cumulative distribution function of the t-distribution

In R: `pt(q, df)`, `qt(p, df)`, `dt(x, df)`

t-test: Summary

Data Setting One sample, no explanatory variable $Y_1, \ldots, Y_n$ i.i.d from population with unknown variance σ^2

Null hypothesis $H_0 : \mu = \mu_0$

Test statistic

$$t(\mu_0) = \frac{\overline{Y} - \mu_0}{\sqrt{s^2/n}}$$

Reference distribution $t(\mu_0) \dot{\sim} t_{(n-1)}$

t-test: Summary

Rejection Region for level α test

One sided $H_A : \mu < \mu_0$	Two sided $H_A : \mu \neq \mu_0$	One sided $H_A : \mu > \mu_0$
$t(\mu_0) < t_{(n-1)\alpha}$	$\lvert t(\mu_0) \rvert > t_{(n-1)1-\alpha/2}$	$t(\mu_0) > t_{(n-1)1-\alpha}$

$t_{(n-1)\alpha} =$ `qt(alpha, df = n - 1)`

t-test: Summary

p-values given an observed $t(\mu_0) = t$

One sided $H_A : \mu < \mu_0$	Two sided $H_A : \mu \neq \mu_0$	One sided $H_A : \mu > \mu_0$
$F_t(t; n-1)$	$2\left(1 - F_t(\lvert t \rvert; n-1)\right)$	$1 - F_t(t; n-1)$

$F_t(t; n-1) =$ `pt(t, df = n-1)`

Confidence Intervals $(1-\alpha)100\%$

$$\left(\overline{Y} - t_{(n-1)1-\alpha/2}\sqrt{\frac{s^2}{n}}, \overline{Y} + t_{(n-1)1-\alpha/2}\sqrt{\frac{s^2}{n}}\right)$$

Standard error

$$Var(\overline{Y}) = \frac{\sigma^2}{n}$$

We estimated σ^2 with s^2, hence estimate $Var(\overline{Y})$ with

$$\widehat{Var}(\overline{Y}) = \frac{s^2}{n}$$

Square root of this, often called, **standard error of the mean**:

$$\text{SE}(\overline{Y}) = \frac{s}{\sqrt{n}}$$

In general **standard error** refers to the *estimated* standard deviation of an estimator.

Next time

Population proportions (a special case of population means)