Tests of proportions

ST551 Lecture 11

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Setup

Population: $Y \sim$ Bernoulli(p), i.e.

$$Y = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p \end{cases}$$

Parameter: $\mu = E(Y) = p$, the population proportion

Sample: n i.i.d from population: $Y_1, \dots, Y_n$

Statistic: $X = \sum_{i=1}^{n} Y_i$, the count of 1's in the sample.

We want to test the hypothesis: $H_0 : p = p_0$

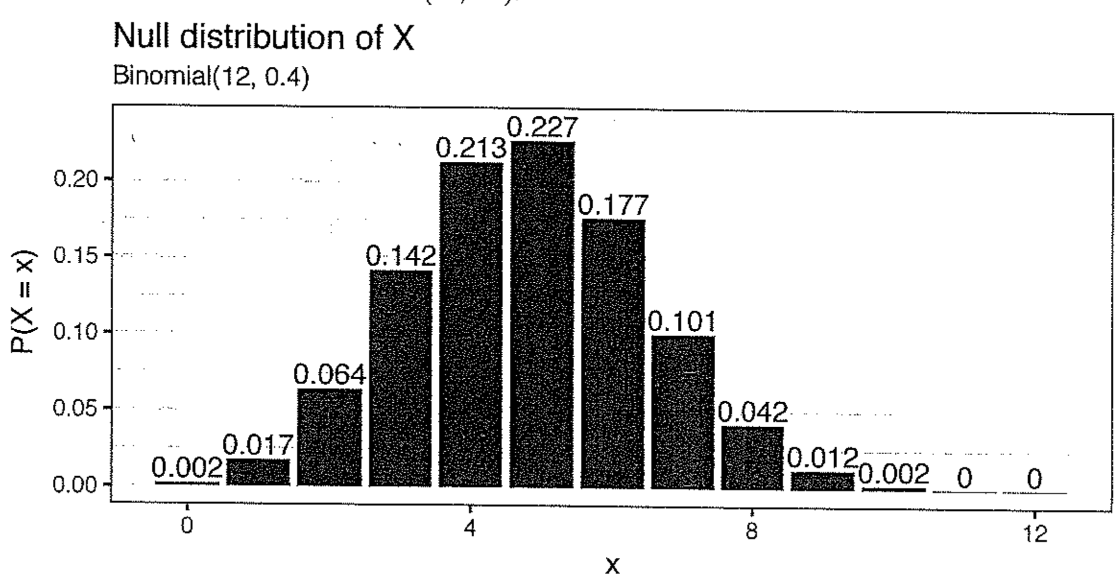
Null distribution: Under the null hypothesis our statistic has a Binomial(n, p_0) distribution.

One sided test

Imagine, $n = 12$, and we are interested in the hypotheses:

$$H_0 : p = 0.4 \quad \text{versus} \quad H_A : p > 0.4$$

The null distribution for X is Binomial$(12, 0.4)$.



Q1: What is the form of the rejection region? (pick one from the below list)

"Consider values of the test statistic which are most unusual and would be more typical if the alternative were true."

- Reject H_0 if $X > c_U$ If $p > 0.4$, expect to see higher counts, i.e. large X
- Reject H_0 if $X < c_L$
- Reject H_0 if $X > c_U$ or $X < c_L$

Q2: What is/are the critical value(s)?

Use the null distribution to find the critical value(s) for one-sided level $\alpha = 0.05$ test.

"Critical values are chosen to obtain the desired significance level."

Using barplot from previous page. Find c s.t. sum of probabilities of ~~8~~ $P(X=k)$ for $k \geq c$ is just below α.

$P(X \geq 11) = P(X=11) + P(X=12) = 0$

$P(X \geq 10) = P(X=10) + P(X \geq 11) = 0.002$

$P(X \geq 9) = P(X=9) + P(X \geq 10) = 0.002 + 0.012 = 0.014 < \alpha$

$P(X \geq 8) = P(X=8) + P(X \geq 9) = 0.014 + 0.042 = 0.056 > \alpha$

Reject Ho when $X \geq 9$ (or equivalently $X > 8$)

Q3: Imagine X = 7, what would the one-sided p-value be?

$P_{H_0}(X \geq 7) = 0.157$

Q4: Can you generalise and write down the form of the rejection region and p-value for $H_0 : p = p_0$ versus $H_A : p > p_0$ when the observed count is X = x?

Reject Ho for $X > c$

Find c by looking for value s.t.

$$P(X > c) \leq \alpha$$

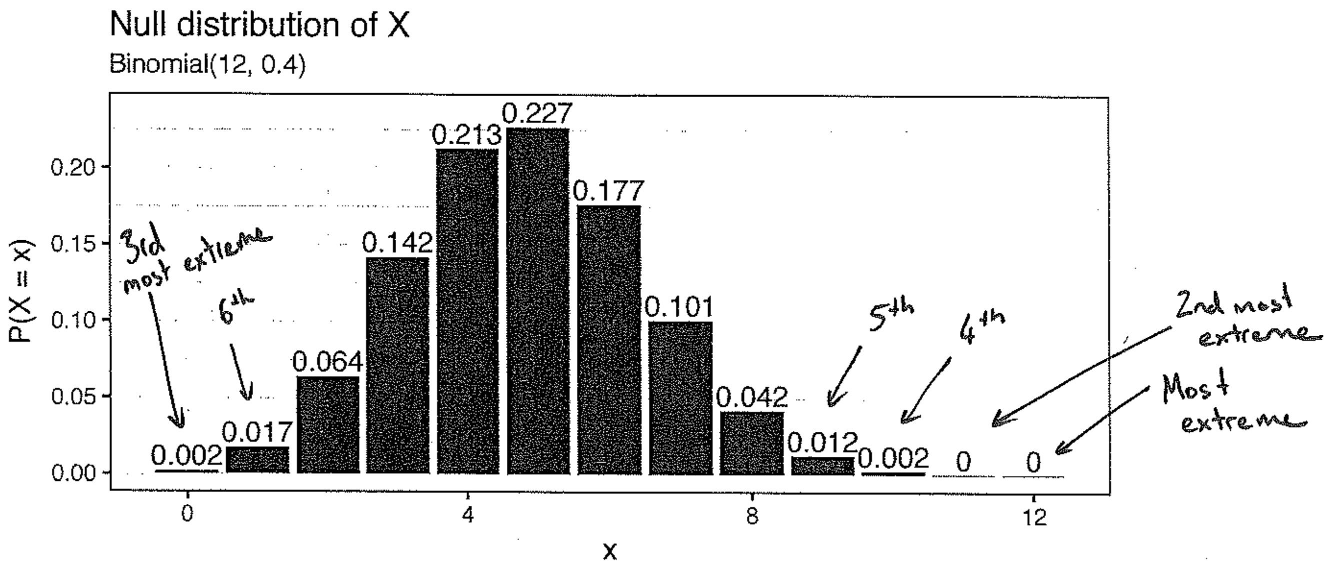
avoids us making Type I error <u>more</u> than α100% of the time, but likely true error rate is $\ll \alpha$.

Two sided Rejection Region

Now consider:

$$H_0 : p = 0.4 \quad \text{versus} \quad H_A : p \neq 0.4$$

The null distribution is the same:



Q5: What is the form of the rejection region?

"Consider values of the test statistic which are most unusual and would be more typical if the alternative were true."

- Reject H_0 if $X > c_U$
- Reject H_0 if $X < c_L$
- Reject H_0 if $X > c_U$ or $X < c_L$

p could be greater than p_0 ⇒ X large
p " " less " " ⇒ X small

Q6: What is/are the critical value(s)?

Use the null distribution to find the critical value(s) for a two-sided level $\alpha = 0.05$ test. *This is tricky, how will you measure which are the most extreme values of X?*

Most extreme = Least probable

Add up probabilities from most extreme to least extreme until sum is just below 0.05.

$P(X=12) + P(X=11) + P(X=0) + P(X=10) + P(X=9) + P(X=1) = 0.033 < \alpha$

Adding one more : $P(X=8)$ would put us over α

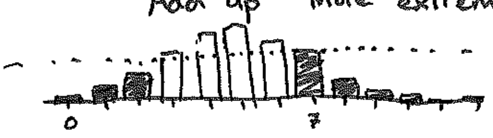
Reject H_0 for $X \leq 1$ or $X \geq 9$

Two sided p-value

Q7: Imagine X = 7, what would the two-sided p-value be?

Add up "more extreme" probabilities. $P(X=7) = 0.101 \Rightarrow$

Add up all $P_{H_0}(X=k) \leq 0.101$



$0.002 + 0.017 + 0.064 + 0.101 + 0.042 + 0.012 + 0.002$
$= 0.2417$

Confidence Interval

95%

Q8: Again, imagine X = 7, use the distributions on the following page to construct a (rough) confidence interval.

"All values of μ_0 that would not be rejected in a **two-sided** hypothesis test of $H_0 : \mu = \mu_0$."

For each $p = p_0$, would we reject $H_0: p = p_0$ when $X = 7$?

two sided

Easiest to ask is p-value for $X = 7 < 0.05$?

Complete on Monday....

Null distributions of X

Binomial(12, p)