Sign-Rank Test

ST551 Lecture 14

Charlotte Wickham

2017-10-20

Wilcoxon Signed Rank Test

Usual setting

Population: $Y \sim$ some population distribution

Sample: n i.i.d from population: $Y_1, \ldots, Y_n$

Parameter: ?

Null Hypothesis the population 'center' is c_0.

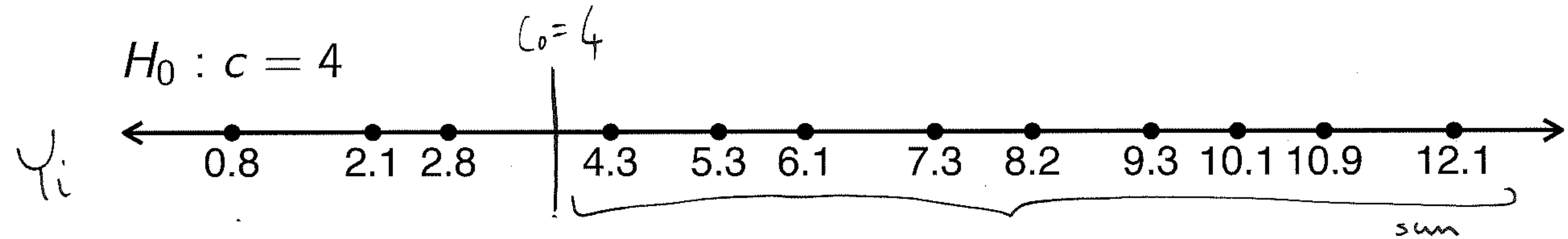
Let's talk about the procedure first, then come back to why it's hard to be specific here.

Wilcoxon Signed Rank Test Procedure

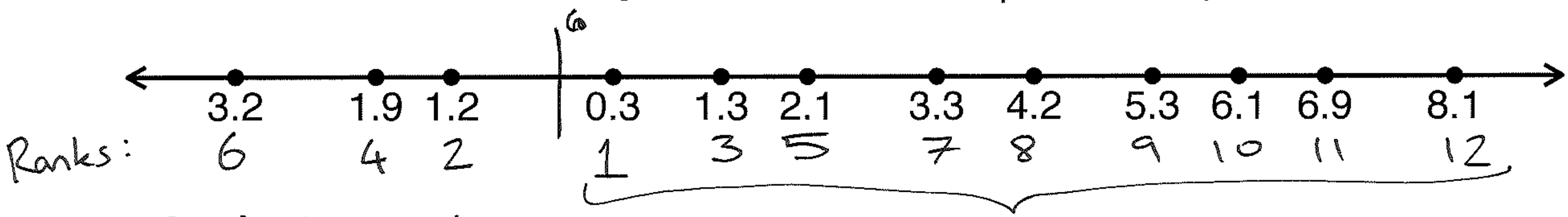
1. Find the distance of each observed value from the hypothesized center, c_0.
2. Assign a rank to each observation based on its distance from c_0: from 1 for closest, to n for furthest from c_0.
3. **Test statistic**: $S =$ Sum of the ranks for the values that were **larger** than c_0.

Example: test statistic calculation

$H_0 : c = 4$



1. Find distance to $c_0 = 4$. $|Y_i - 4|$



2. Assign ranks
3. **Test statistic:** $S =$ sum of ranks for $Y_i > 4 =$ 1+3+5+7+8 +9+10+11+12

$S = 66$

Reference distribution

Either:

- Use an exact p-value, by assuming *each rank* has the same chance of being assigned above or below c_0, **or**
- Use the Normal approximation to the null distribution of S

Reference distribution: Exact p-values

If the population distribution were symmetric about c_0, each rank $1, \ldots, n$ independently has probability 0.5 of being assigned to an observation above c_0.

We can consider all possible ways of assigning the ranks $1, \ldots, n$ above and below c_0 to work out the exact reference distribution (this is what the R function `wilcox.test()` does if you use the argument `exact = TRUE`)

Reference distribution: Normal approximation p-values

If the population distribution were symmetric about c_0,

$$E(S) = \frac{n(n+1)}{4}, \quad Var(S) = \frac{n(n+1)(2n+1)}{24}$$

↑ statistic

(*Can prove by considering S as a sum of products between Bernoulli(0.5) r.v's and the integers $1, \ldots, n$*)

So, we can construct a Z-statistic

$$Z = \frac{S - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}} \leftarrow \sqrt{\frac{Var(Y)}{n}}$$

and compare it to a N(0, 1)

Example: continued

$$E(S) = \frac{n(n+1)}{4} = \frac{12(13)}{4} = 39$$

$$Var(S) = \frac{n(n+1)(2n+1)}{24} = \frac{12(13)(25)}{24} = 162.5$$

$$Z = \frac{66 - 39}{\sqrt{162.5}} = 2.12$$

$\alpha = 0.05$
two sided

$z_{1-\alpha/2} = 1.96$

```
2 * (1 - pnorm(abs(z)))
```

```
## [1] 0.03417047
```

p-value

Why is it hard to say what it tests?

Your turn: Sketch worksheet

Performance of the Wilcoxon Signed Rank Test

With no additional assumptions

As a test of the population mean:

- The Wilcoxon Signed Rank test **is not assymptotically exact**
- The Wilcoxon Signed Rank test **is not consistent**

As a test of the population median:

- The Wilcoxon Signed Rank test **is not assymptotically exact**
- The Wilcoxon Signed Rank test **is not consistent**

Performance of the Wilcoxon Signed Rank Test

If you add an assumption: the population distribution is symmetric.

The Wilcoxon Signed Rank test **is assymptotically exact**

The Wilcoxon Signed Rank test **is consistent**

Null hypothesis: $\mu = M = c_0$

We learn about the mean/median. Of course we could learn more about these parameters directly with a t-test or sign test without the additional symmetry assumption.

In R

```
y <- c(0.8, 2.1, 2.8, 4.3, 5.3, 6.1, 7.3, 8.2,
  9.3, 10.1, 10.9, 12.1)
```

Exact p-values with `exact = TRUE` (default)

```
wilcox.test(y, mu = 4, exact = TRUE)
```

```
##
##  Wilcoxon signed rank test
##
## data:  y
## V = 66, p-value = 0.03418
## alternative hypothesis: true location is not equal to 4
```

In R

```
y <- c(0.8, 2.1, 2.8, 4.3, 5.3, 6.1, 7.3, 8.2,
  9.3, 10.1, 10.9, 12.1)
```

Approximate p-values with `exact = FALSE` and no continuity correction

```
wilcox.test(y, mu = 4, exact = FALSE, correct = FALSE)
```

```
##
##  Wilcoxon signed rank test
##
## data:  y
## V = 66, p-value = 0.03417
## alternative hypothesis: true location is not equal to 4
```