Tests of scale

ST551 Lecture 15

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So far . . .

Tests of *center*

Test/Procedure	Parameter	Setting
Z-test	Population mean	
t-test	Population mean	
Binomial exact test	Population proportion (mean)	
Binomial z-test	Population proportion (mean)	
Sign test	Population median	
Signed Rank test	Population mean/median	

Two tests of scale

1. Chi-square test of variance
2. t-test of variance

Chi-square test of variance

Chi-square test of variance

Population: $Y \sim$ some population distribution

Sample: n i.i.d from population, $Y_1, \ldots, Y_n$

Parameter: Population variance $\sigma^2 = Var(Y)$

Sample variance

The sample variance,

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2$$

is an unbiased and consistent estimate of σ^2.

Furthermore, if $Y \sim N(\mu, \sigma)$, it can be shown the sampling distribution of s^2 is a scaled Chi-square distribution:

$$(n-1)\frac{s^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

Using the sampling distribution to formulate a test

Assume the population distribution is $N(\mu, \sigma^2)$.

Consider the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$.

Let the test statistic be:

$$X(\sigma_0^2) = (n-1)\frac{s^2}{\sigma_0^2}$$

What's the distribution of the test statistic if the null hypothesis is true?

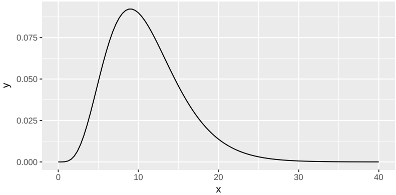
Rejection regions

For a test at level α:

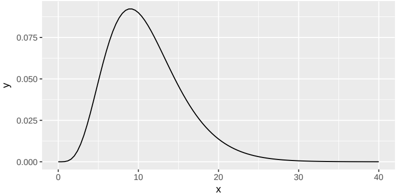
- $H_A : \sigma^2 > \sigma_0^2$: Reject H_0 if $X(\sigma_0^2) >$
- $H_A : \sigma^2 < \sigma_0^2$: Reject H_0 if $X(\sigma_0^2) <$
- $H_A : \sigma^2 \neq \sigma_0^2$: Reject H_0 if $X(\sigma_0^2) >$ or $X(\sigma_0^2) <$

p-values: Your turn

Shade the area for the p-value when $X(\sigma_0) = 20$, with $H_A : \sigma^2 > \sigma_0^2$

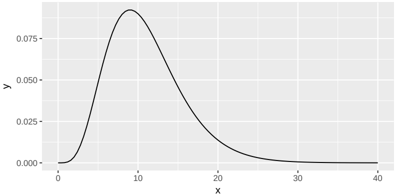


Shade the area for the p-value when $X(\sigma_0) = 20$, with $H_A : \sigma^2 < \sigma_0^2$



p-values: Your turn

Shade the area for the p-value when $X(\sigma_0) = 20$, with $H_A : \sigma^2 \neq \sigma_0^2$



p-values: In general

p-value is:

- $H_A : \sigma^2 > \sigma_0^2$:
- $H_A : \sigma^2 < \sigma_0^2$:
- $H_A : \sigma^2 \neq \sigma_0^2$:

where $X \sim \chi^2_{(n-1)}$

In R: $P(X \leq x) =$ `pchisq(x, df = n - 1)`

Confidence interval

From inverting test statistic.

$(1-\alpha)100\%$ confidence interval

$$\left(\frac{s^2(n-1)}{\chi^2_{(n-1)}(1-\alpha/2)}, \frac{s^2(n-1)}{\chi^2_{(n-1)}(\alpha/2)} \right)$$

What if the population isn't Normal

(from Sarah Emerson's slides F2016)

Simulated Rejection Rates under H_0

(nominal level $\alpha = 0.05$)

Population Distribution	True Population Variance	$n = 50$	$n = 500$	$n = 5000$
t_5	5/3	0.1208	0.1481	0.1748
χ_{10}	20	0.0903	0.0970	0.0973
Exponential(1)	1	0.1630	0.1943	0.2033
Uniform(0, 1)	1/12	0.0048	0.0046	0.0059

Figure 1

t-test of variance

t-test of variance

An alternative to the Chi-square test of variance, based on considering a transformed response:

$$Z_i = \left(Y_i - \overline{Y}\right)^2 \quad i = 1, \ldots, n$$

What is $E(Z)$, are the Z_i independent?

A CLT for the sample variance

The Z_i aren't independent but are **weakly** dependent, turns out there is a CLT for this case, as long as Z has finite fourth moment:

$$\frac{\overline{Z} - E(Z)}{\sqrt{Var(Z)/n}} \rightarrow_d N(0,1)$$

Substitute in for $E(Z)$

$$\frac{\overline{Z} - \frac{n-1}{n}\sigma^2}{\sqrt{Var(Z)/n}} \rightarrow_d N(0,1)$$

Leads to a t-test

We don't know the population variance of the Z (transformed Y), so substitute sample estimate for it.

Under null hypothesis $H_0 : \sigma^2 = \sigma_0^2$

$$t(\sigma_0^2) = \frac{\overline{Z} - \frac{n-1}{n}\sigma_0^2}{\sqrt{s_Z^2/n}} \dot{\sim} t_{(n-1)}$$

So to test the null $H_0 : \sigma^2 = \sigma_0^2$, do a t-test on $Z_i = \left(Y_i - \overline{Y}\right)^2$ with the null hypothesis $H_0 : \mu_Z = \frac{n-1}{n}\sigma_0^2$.

Performance of t-test of variance

Charlotte's simulations, rejection rate of H_0 for $\alpha = 0.05$.

	50	500	5000
Chi-square(10)	0.12	0.069	0.055
Exp(1)	0.177	0.084	0.072
t(5)	0.148	0.076	0.068
Uniform(0, 1)	0.057	0.057	0.042