Tests of scale

ST551 Lecture 15

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So far . . .

Tests of *center*

Test/Procedure	Parameter	Setting
Z-test	Population mean	Sample i.i.d. of size n, know σ^2
t-test	Population mean	" " unknown σ^2
Binomial exact test	Population proportion (mean)	Population Bernoulli(p) exact sampling dist of $\sum Y_i$
Binomial z-test	Population proportion (mean)	approximation $\Longrightarrow$ large n
Sign test	Population median	Sample i.i.d. of size n
Signed Rank test	Population mean/median	Sample i.i.d. of size n population distribution is symmetric

$n \to \infty$ equivalent

If population is truely symmetric Signed Rank test may have better power than Sign test.

More assumptions, if they are true, often leads to higher power.

<u>Midterm</u>

p-value
critical value

No statistical tables

- To write in R code or notation the value you want
- Give you some values
Choose the right one

Two tests of scale

spread

1. Chi-square test of variance
2. t-test of variance

Chi-square test of variance

Chi-square test of variance

Population: $Y \sim$ some population distribution

Sample: n i.i.d from population, $Y_1, \ldots, Y_n$

Parameter: Population variance $\sigma^2 = Var(Y)$

Sample variance

The sample variance,

$$\hat{\sigma}^2 = \quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}\right)^2$$

is an unbiased and consistent estimate of σ^2.

Furthermore, if $Y \sim N(\mu, \sigma)$, it can be shown the sampling distribution of s^2 is a scaled Chi-square distribution:

$$(n-1)\frac{s^2}{\sigma^2} \sim \chi^2_{\underbrace{(n-1)}_{\text{degrees of freedom parameter}}}$$

Using the sampling distribution to formulate a test

Assume the population distribution is $N(\mu, \sigma^2)$.

Consider the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$.

if the ratio $\frac{s^2}{\sigma_0^2} > 1$

evidence in favor of $\sigma^2 > \sigma_0^2$

Let the test statistic be:

$$X(\sigma_0^2) = (n-1)\frac{s^2}{\sigma_0^2}$$

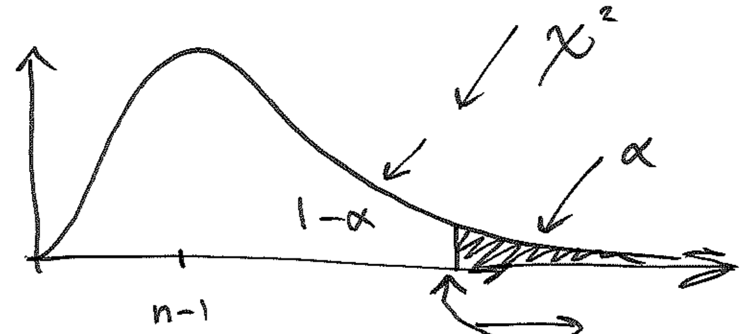
sample variance

hypothesized population variance

What's the distribution of the test statistic if the null hypothesis is true?

Under H_0 $\quad X(\sigma_0^2) \sim \chi_{(n-1)}$

Rejection regions



For a test at level α:

- $H_A : \sigma^2 > \sigma_0^2$: Reject H_0 if $X(\sigma_0^2) >$ $\chi^2_{(n-1)}(1-\alpha)$
- $H_A : \sigma^2 < \sigma_0^2$: Reject H_0 if $X(\sigma_0^2) <$ $\chi^2_{(n-1)}(\alpha)$
- $H_A : \sigma^2 \neq \sigma_0^2$: Reject H_0 if $X(\sigma_0^2) >$ $\chi^2_{(n-1)}(1-\alpha/2)$ or $X(\sigma_0^2) <$ $\chi^2_{(n-1)}(\alpha/2)$

find critical value

In R:

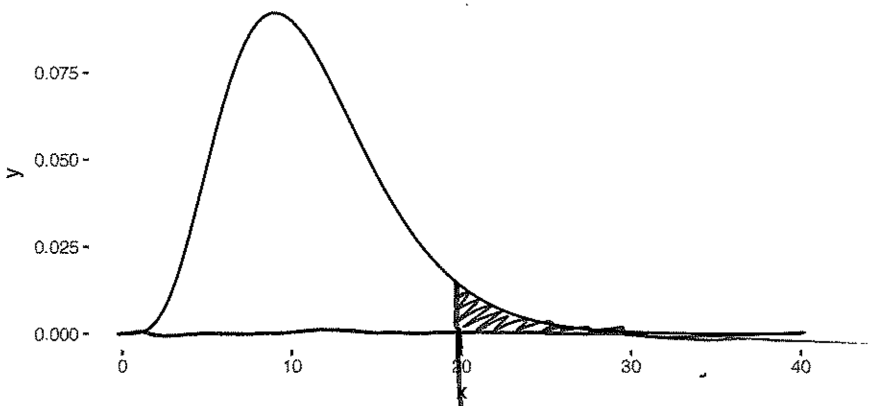
qchisq(1−α, df = n−1)

p-values: Your turn

n = 12

Shade the area for the p-value when $X(\sigma_0) = 20$, with $H_A: \sigma^2 > \sigma_0^2$

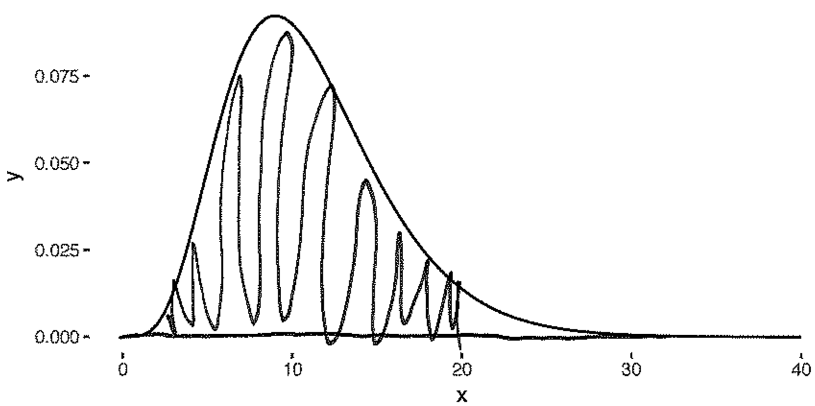
$\chi^2_{(11)}$ if the null is true



obs. $X(\sigma_0)$

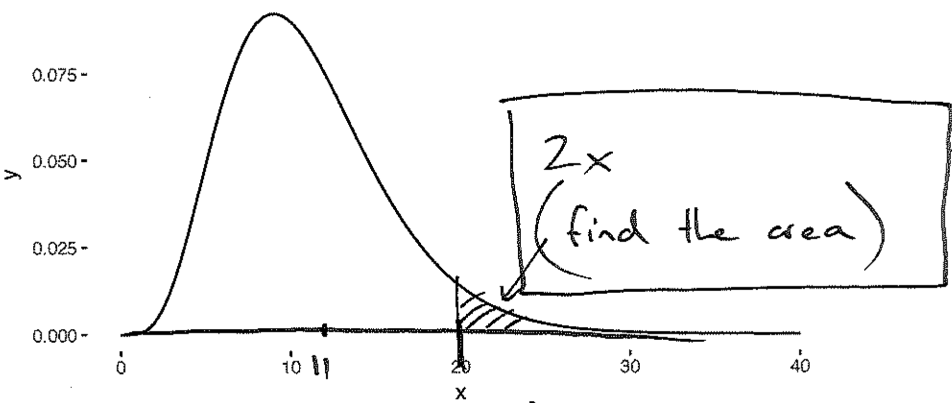
p-value is the probability, if the null is true, of seeing a test statistic as or more extreme than the one observed.

Shade the area for the p-value when $X(\sigma_0) = 20$, with $H_A: \sigma^2 < \sigma_0^2$



p-values: Your turn

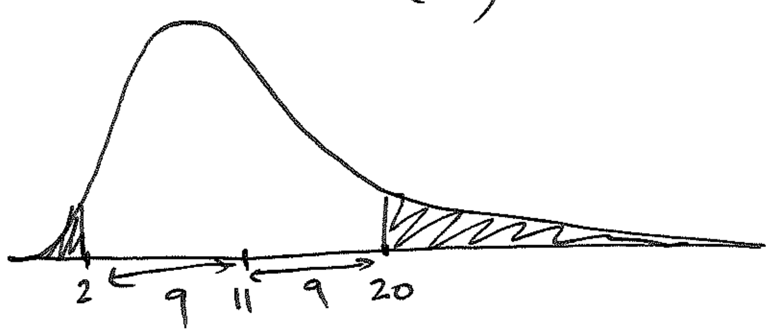
Shade the area for the p-value when $X(\sigma_0) = 20$, with $H_A : \sigma^2 \neq \sigma_0^2$

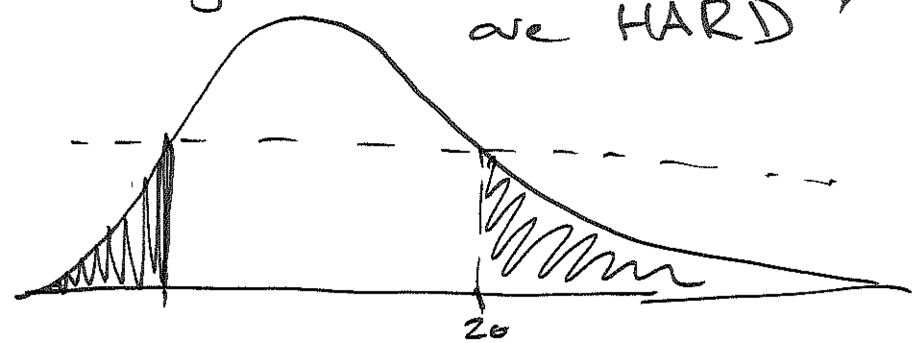


the one used for this Chisquare variance test

"2-sided p-values for assymetric reference dist. are HARD"

obs. $X(\sigma_0)$





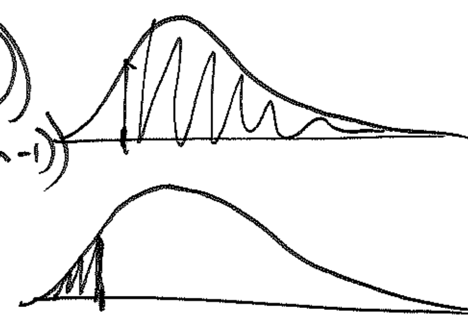
p-values: In general

p-value is:

obs. test stat

- $H_A : \sigma^2 > \sigma_0^2$: $\cancel{P(X(\sigma_0) >}$ $P(X \geq X(\sigma_0^2))$
- $H_A : \sigma^2 < \sigma_0^2$: $P(X \leq X(\sigma_0^2))$
- $H_A : \sigma^2 \neq \sigma_0^2$: $2 * \min(P(X \geq X(\sigma_0^2)), P(X < X(\sigma_0^2)))$

2 * min(pchisq(x, n-1), 1 - pchisq(x, n-1))



where $X \sim \chi^2_{(n-1)}$

In R: $P(X \leq x) =$ `pchisq(x, df = n - 1)`

Confidence interval

From inverting test statistic.

$(1-\alpha)100\%$ confidence interval

$$\left(\frac{s^2(n-1)}{\chi^2_{(n-1)}(1-\alpha/2)}, \frac{s^2(n-1)}{\chi^2_{(n-1)}(\alpha/2)}\right)$$

What if the population isn't Normal

(from Sarah Emerson's slides F2016)

Simulated Rejection Rates under H_0 (nominal level $\alpha = 0.05$)

Population Distribution	True Population Variance	$n = 50$	$n = 500$	$n = 5000$
t_5	5/3	0.1208	0.1481	0.1748
χ_{10}	20	0.0903	0.0970	0.0973
Exponential(1)	1	0.1630	0.1943	0.2033
Uniform(0, 1)	1/12	0.0048	0.0046	0.0059

Figure 1

The Normality assumption is crucial for good performance

t-test of variance

t-test of variance

$E[Y] = \mu$
$Var[Y] = \sigma^2$

setting is the same

An alternative to the Chi-square test of variance, based on considering a transformed response:

$\frac{1}{n}\sum_{i=1}^{n} Z_i$

$\downarrow$

look a little like s^2

$$Z_i = \left(Y_i - \overline{Y}\right)^2 \quad i = 1, \ldots, n$$

What is $E(Z)$, are the Z_i independent?

$Var(X) = E(X^2) - (E(X))^2$

$Var(\bar{Y}) = E(\bar{Y}^2) - (E(\bar{Y}))^2$

$\frac{\sigma^2}{n} = E(\bar{Y}^2) - \mu^2$

$$\begin{aligned}
E(Z_i) &= E\left(\left(Y_i - \bar{Y}\right)^2\right) \\
&= E\left(Y_i^2 - 2Y_i\bar{Y} + \bar{Y}^2\right) \\
&= E(Y_i^2) - 2E(Y_i\bar{Y}) + E(\bar{Y}^2) \\
&= \mu^2 + \sigma^2 - 2E(Y_i\bar{Y}) + \frac{\sigma^2}{n} + \mu^2
\end{aligned}$$

you could show ...

$$= \sigma^2 \frac{n-1}{n}$$

A CLT for the sample variance

The Z_i aren't independend but are **weakly** dependent, turns out there is a CLT for this case, as long as Z has ~~a~~ finite fourth moment:

$$\frac{\overline{Z} - E(Z)}{\sqrt{Var(Z)/n}} \rightarrow_d N(0,1)$$

Substitute in for $E(Z)$

$$\frac{\overline{Z} - \frac{n-1}{n}\sigma^2}{\sqrt{Var(Z)/n}} \rightarrow_d N(0,1)$$

sample mean of Z_i's

$\frac{n-1}{n} s^2$

Leads to a t-test

We don't know the population variance of the Z (transformed Y), so substitute sample estimate for it.

Under null hypothesis $H_0 : \sigma^2 = \sigma_0^2$

$$t(\sigma_0^2) = \frac{\overline{Z} - \frac{n-1}{n}\sigma_0^2}{\sqrt{s_Z^2/n}} \dot{\sim} t_{(n-1)}$$

sample variance of Z_i's

$\frac{1}{n}\sum_{i=1}^{n} Z_i = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$

$= \frac{n-1}{n-1}\frac{1}{n}\sum_{i=1}^{n}(Y_i - \overline{Y})^2$

$= \frac{n-1}{n} s^2$

(1) So to test the null $H_0 : \sigma^2 = \sigma_0^2$, do a t-test on $Z_i = \left(Y_i - \overline{Y}\right)^2$ with the null hypothesis $H_0 : \mu_Z = \frac{n-1}{n}\sigma_0^2$.

(2) Why use it instead of χ^2 test?

If population is Normal, χ^2 test is good, but otherwise performs poorly!

Doesn't matter what the population is t-test rapidly approaches rejection rate of α under Null