Kolmogorov-Smirnov Test (not on midterm!)

Kolmogorov-Smirnov Test

Population: $Y \sim$ some population distribution with c.d.f F

Sample: n i.i.d from population, $Y_1, \ldots, Y_n$

Parameter: Whole CDF

Null hypothesis: $H_0 : F = F_0$, versus $H_A : F \neq F_0$

c.d.f completely specified

e.g. $N(0, 1)$

Kolmogorov-Smirnov Test

Test statistic

$$D(F_0) = \sup_y \left| \hat{F}(y) - F_0(y) \right|$$

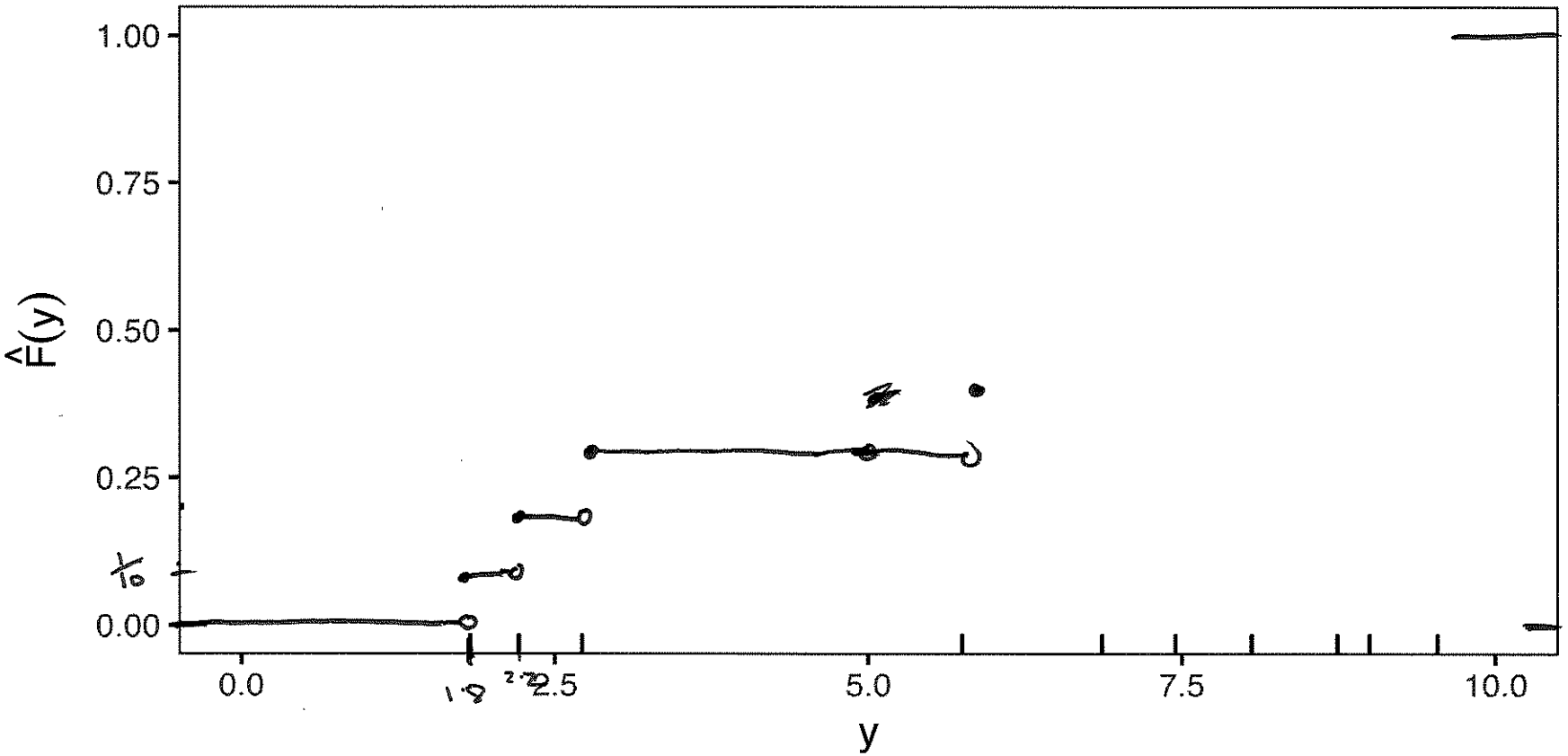
where $\hat{F}(y)$ is the empirical cumulative distribution function:

$$\hat{F}(y) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{Y_i \leq y\}$$

and F_0 is the cumulative distribution function for the null hypothesized distribution.

ECDF: Example

Sample values: *1.8*, *2.2*, *2.7*, *5.7*, *6.9*, *7.4*, *8.1*, *8.7*, *9* and *9.5* $n = 10$

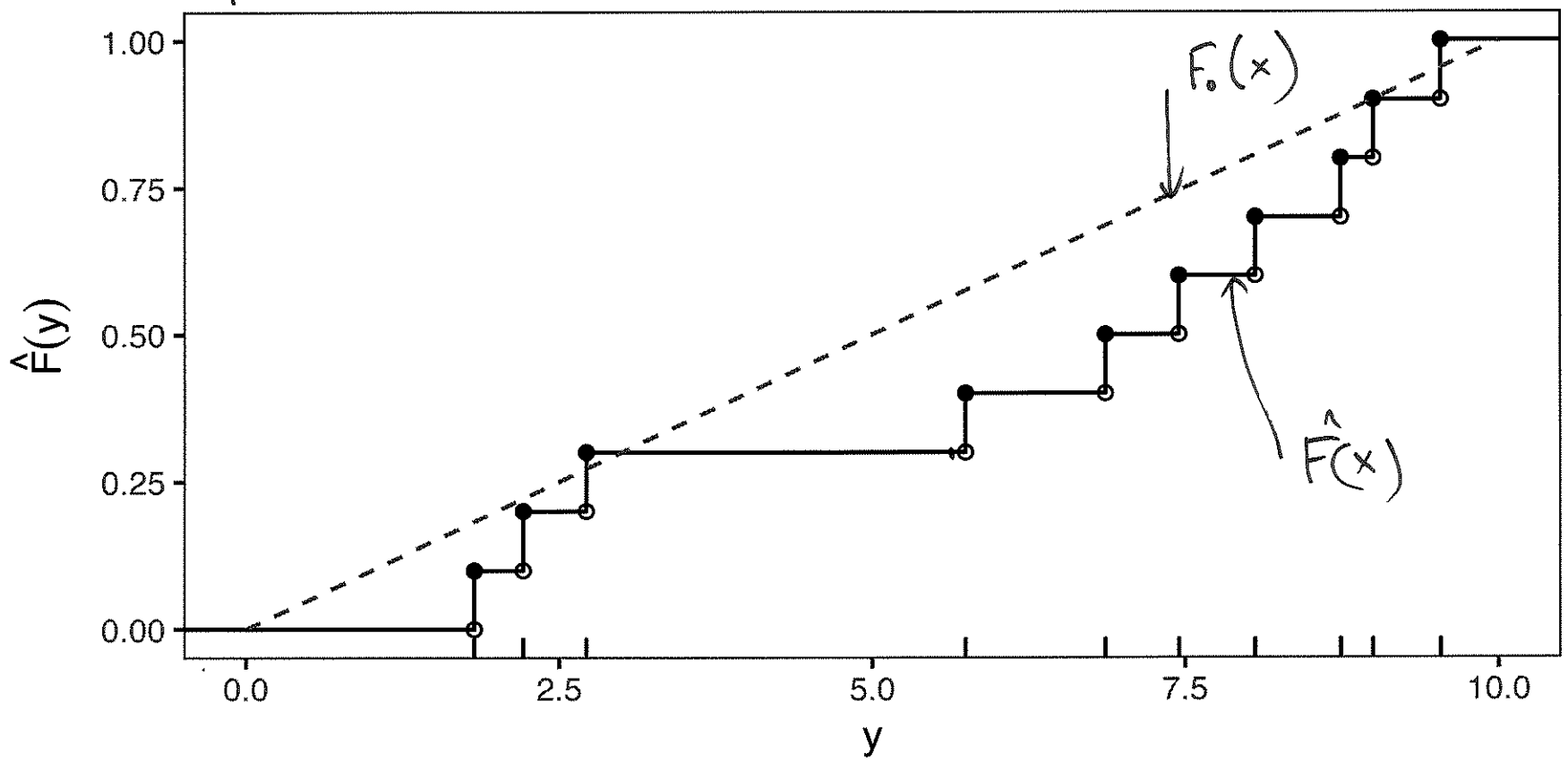


KS test statistic: Uniform(0, 10)

Say,

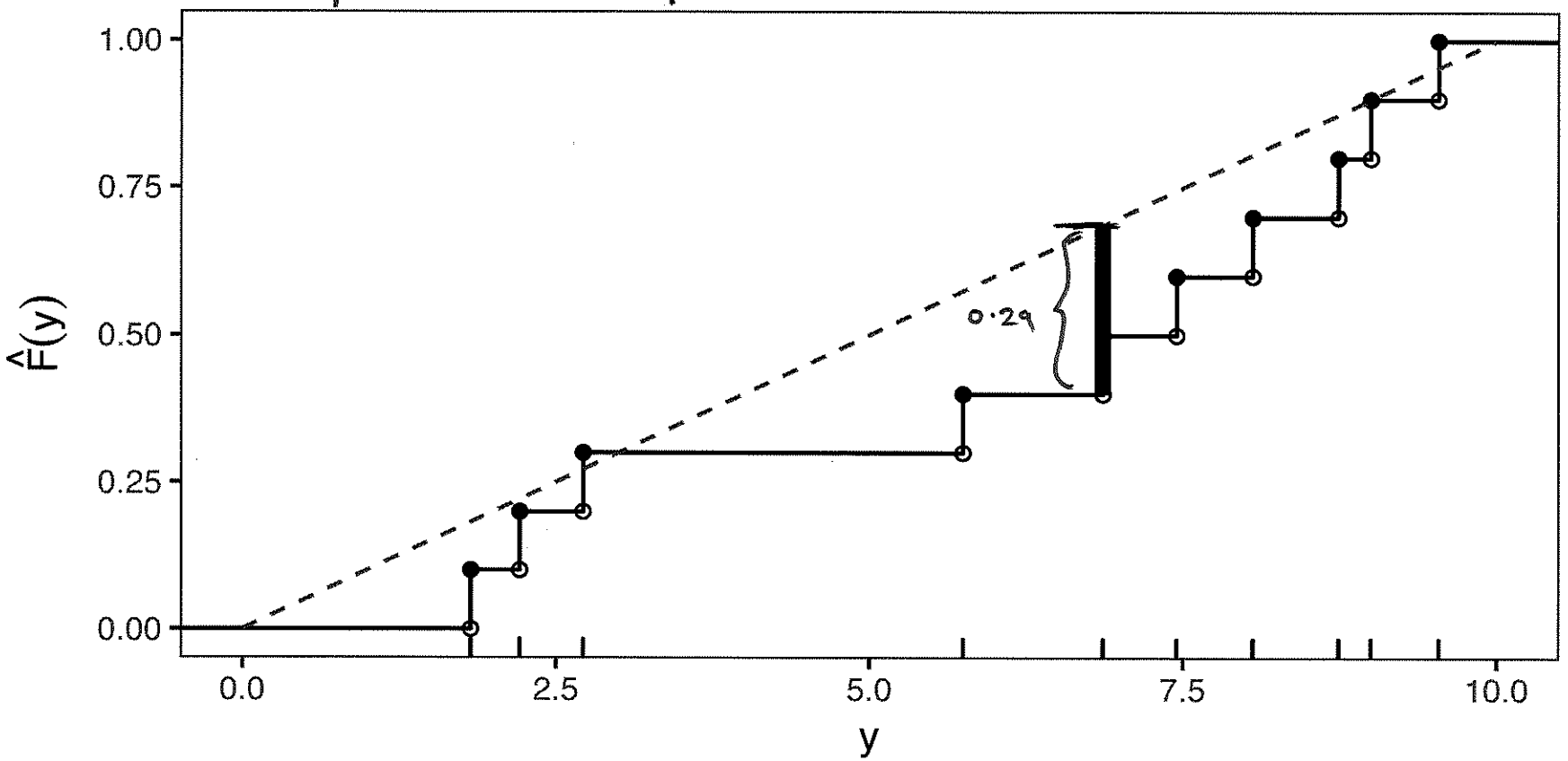
$$H_0 : F(Y) = \begin{cases} y/10, & 0 < y \leq 10 \\ 0, & \text{otherwise} \quad y < 0 \\ 1, & \qquad\qquad y > 10 \end{cases}$$

I.e. $H_0 : Y \sim \text{Uniform}(0, 10)$



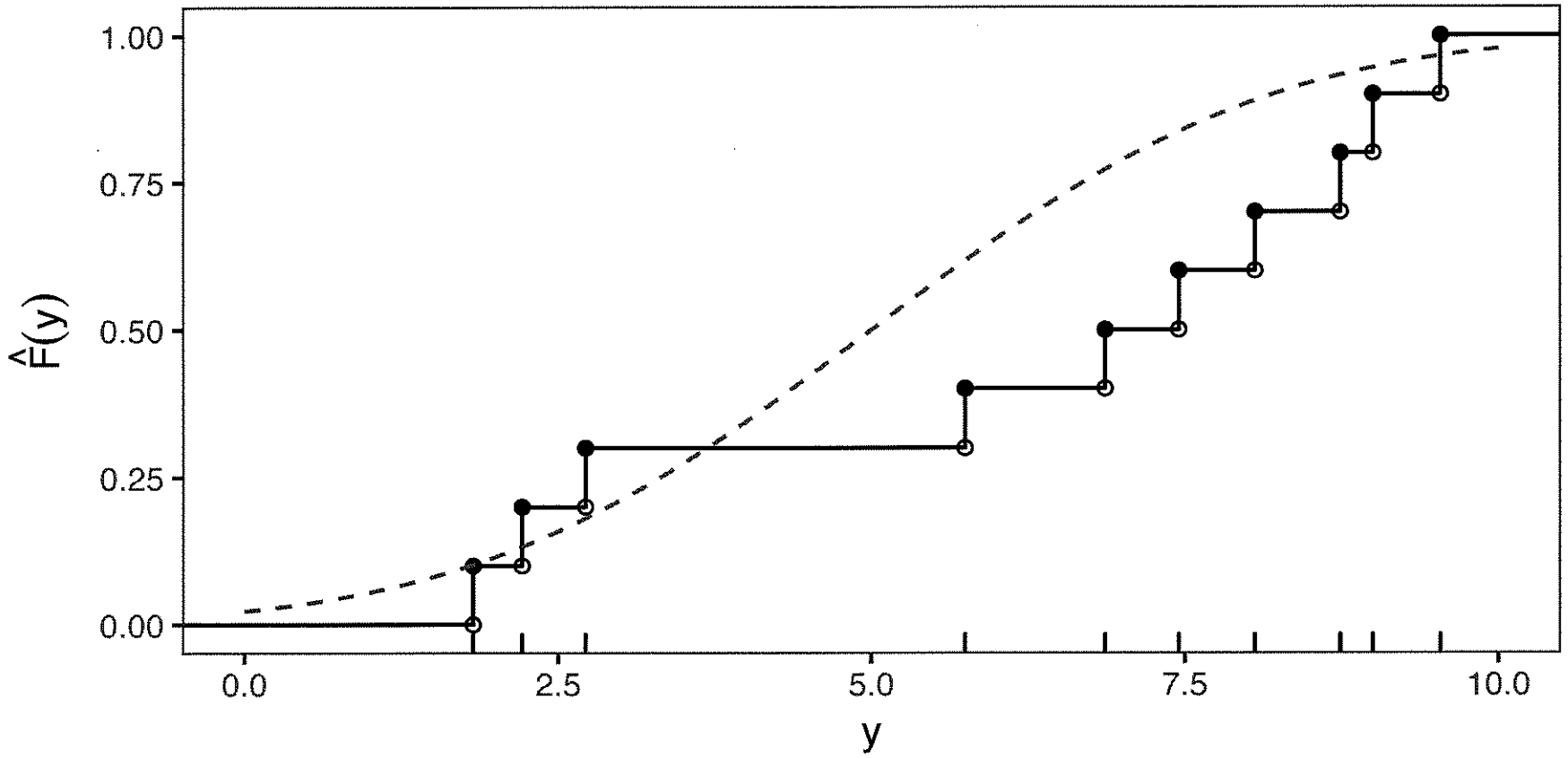
KS test statistic: Uniform(0, 10) cont.

$D(F_0) = \sup_y \left|\hat{F}(y) - F_0(y)\right| \approx 0.29$ (occurs at y just less than 6.9)



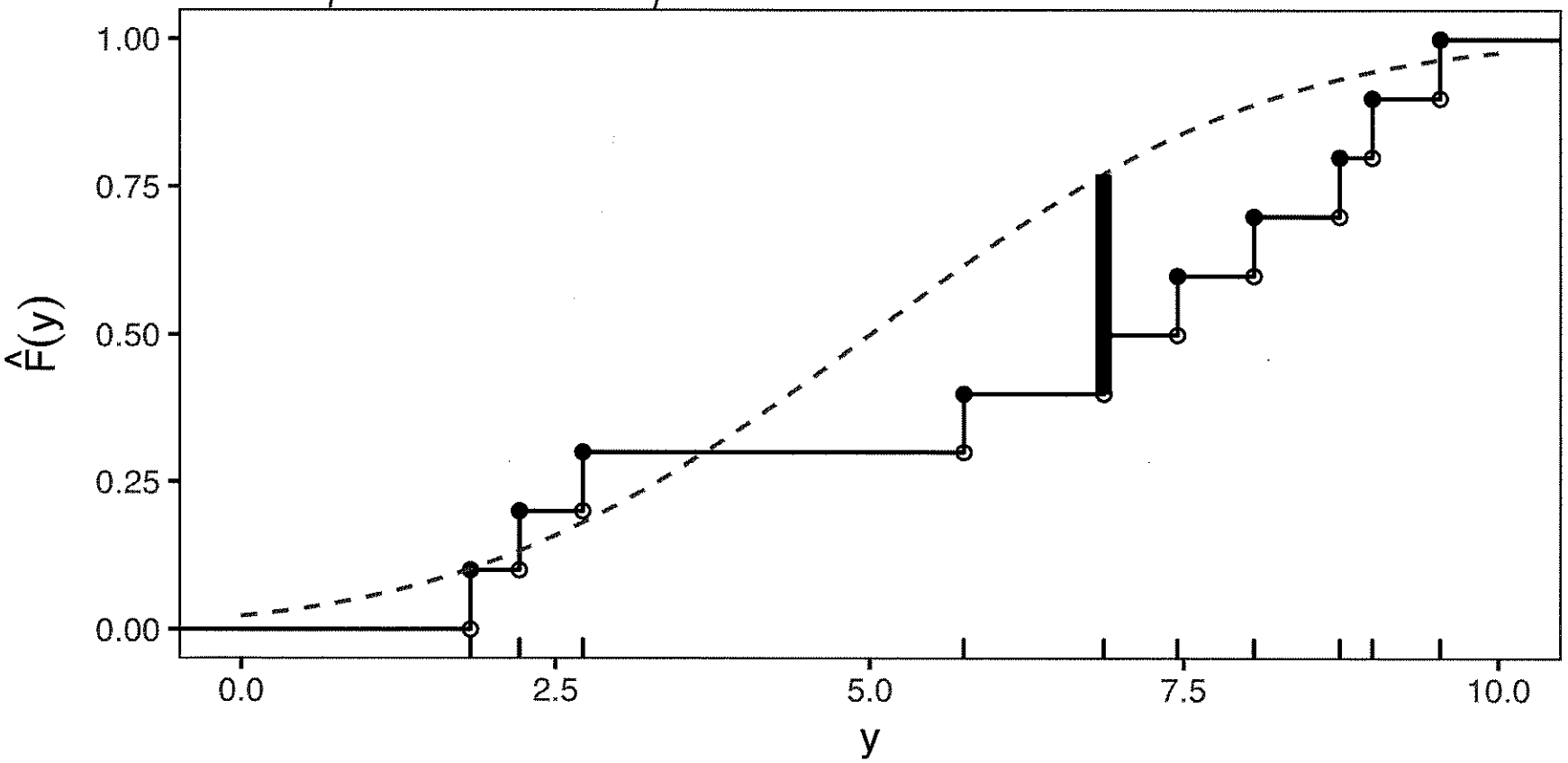
KS test statistic: Normal(5, 6.25)

$H_0 : Y \sim \text{Normal}(5, 6.25)$

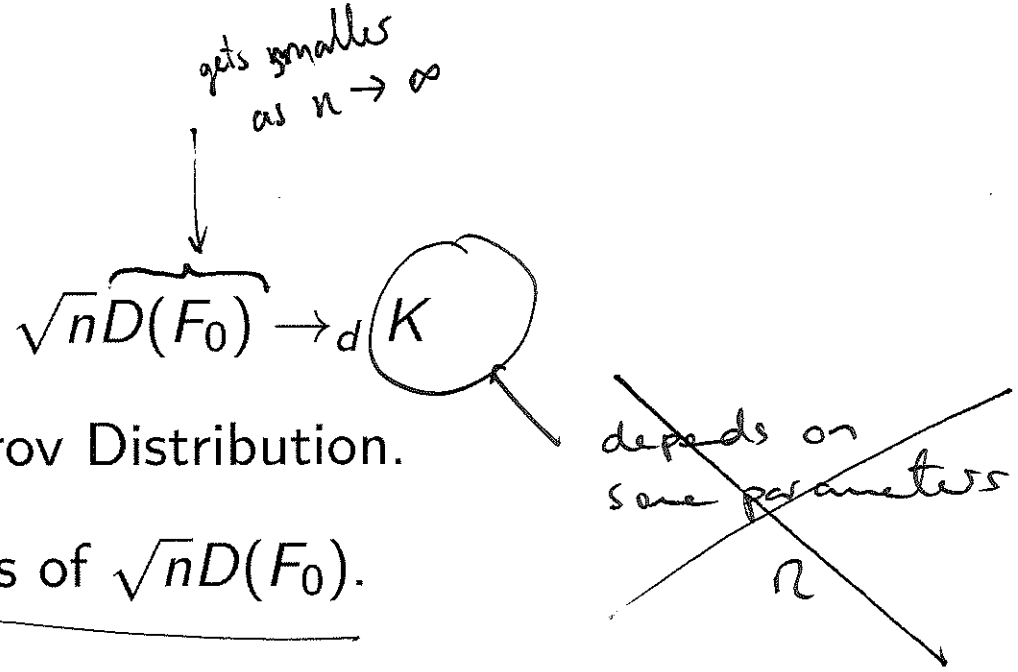


KS test statistic: Example cont.

$D(F_0) = \sup_y \left|\hat{F}(y) - F_0(y)\right| \approx 0.37$ (occurs at y just less than 6.9)



Reference Distribution?



where K is the Kolmogorov Distribution.

Reject H_0 for large values of $\sqrt{n}D(F_0)$.

In R

other arguments

`...` passed to function specified in y

```
ks.test(x = y, y = punif, min = 0, max = 10)
```

vector of obs. values

function null c.d.f.

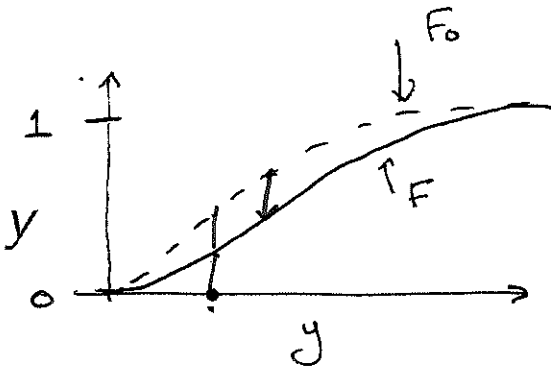
```
##
##  One-sample Kolmogorov-Smirnov test
##
## data:  y
## D = 0.28632, p-value = 0.3209
## alternative hypothesis: two-sided
```

One sided tests

Lesser alternative:

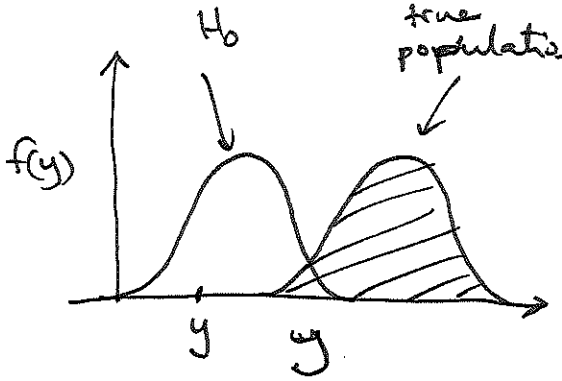
$$H_A : F < F_0, \text{ i.e. } F(y) < F_0(y) \text{ for all } y$$

(F: true c.d.f, F_0: hypothesized c.d.f.)



Test statistic

$$D^-(H_0) = \sup_y (F_0(y) - \hat{F}(y))$$



Greater alternative:

$$H_A : F > F_0, \text{ i.e. } F(y) > F_0(y) \text{ for all } y$$

Test statistic

$$D^+(H_0) = \sup_y (\hat{F}(y) - F_0(y))$$

One sided tests are hard to interpret

Example based on simulated data. $H_0 : Y \sim N(0, 100)$

```
n <- 20
y <- rnorm(n, 0, 1)
```

← true pop. $\neq f(y)$

For greater alternative: $H_A : F_Y(y) > \Phi(y; 0, 100)$ where $\Phi(y; \mu, \sigma^2)$ is the c.d.f of the Normal(μ, σ). $H_0 : F_y(y) = \Phi(y; \mu=0, \sigma^2=100)$

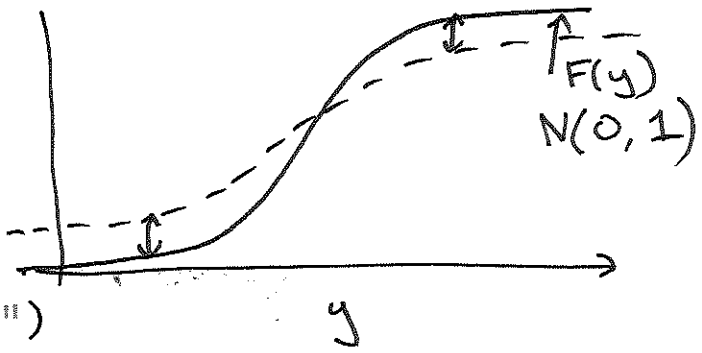
```
ks.test(y, pnorm, 0, 10, alternative = "greater")
```

```
##
##  One-sample Kolmogorov-Smirnov test
##
## data:  y
## D^+ = 0.42016, p-value = 0.000513
## alternative hypothesis: the CDF of x lies above the null hypothesis
```

reject H_0 in favor of H_A:

One sided tests are hard to interpret

For lower alternative: $H_A : F_Y(y) < \Phi(y; 0, 100)$:




```
ks.test(y, pnorm, 0, 10, alternative = "less")

##
##  One-sample Kolmogorov-Smirnov test
##
## data:  y
## D^- = 0.44858, p-value = 0.0001717
## alternative hypothesis: the CDF of x lies below the null hypothesis
```

Reject H_0 in favor of H_A

One sided tests

t-test : $H_A : \mu < \mu_0$

$H_A : \mu > \mu_0$

$H_0 : \mu = \mu_0$

K-S test : $H_A : F(y) < F_0(y)$ for all y

$H_A : F(y) > F_0(y)$

$H_0 : F(y) = F_0(y)$

The combination of the two one-sided alternatives, **does not cover** all the possibilities for which the null hypothesis is false.

This makes it very hard to interpret one-sided KS tests - i.e. don't do a one-sided test.

OR Add assumption → we'll see when we talk about two sample KS test

Estimating parameters

The KS test should only be used if you can completely specify F_0, the population distribution under the null hypothesis.

You should not estimate parameters from the data then do the test.

Kind of like trying to test $H_0 : \mu = \overline{Y}$, you'll rarely reject.

Next time...

After midterm: what if distribution is discrete?