Paired Binary Data

ST551 Lecture 23

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Finish last time's slides

response : Y did you eat breakfast ? yes/no 0/1

groups : G do you ~~prefer~~ cats or dogs? cat / dog 0/1

Paired setting : find n siblings

whose one prefers cats

and one prefers dogs ?

Did you eat breakfast this morning?

Paired Binary Data

Paired Binary Data

Imagine now that our two samples of Bernoulli populations aren't independent, but paired in some way.

$$Y_i, \ldots, Y_n \sim \text{Bernoulli}(p_Y)$$
$$X_i, \ldots, X_n \sim \text{Bernoulli}(p_X)$$

but (Y_i, X_i) are paired.

Examples:

- n subjects with a disease, and n without a disease are sampled then matched (based on demographic factors), response is presence of some risk factor
- Sibling (or twin) studies: n pairs of related people where one falls in one group, and the other falls in the other group, observe some binary response on every person.
- Binary before and after measurements on the same person

Paired Binary Data

Gather sample of $n = 40$ voters.

Before debate: Will you vote for candidate A?
After debate: Will you vote for candidate A?

subject	before	after
1	1	1
2	1	0
3	1	0
4	1	1
5	1	1
6	1	0
⋮		
40		

Just a 2×2 table?

No! ignores the pairing

Group?

Vote for candidate A

	0	1
after	21	19
before	23	17

80

this isn't really 80 independent observations

after

	0	1
0	a = 12	b = 11
1	c = 9	d = 8

before

$40 = n = \#$ pairs

Reflects pairing

→ each pair is counted once

How to analyse?

Option 1: Treat like paired two sample data and do a paired t-test

Option 2: McNemar's test

Paired t-test

$\mu_{before} = \mu_{after}$

Null hypothesis: $H_0 : p_{\text{before}} = p_{\text{after}}$

Look at (per voter) differences:

subject	before	after	diff
1	1	1	0
2	1	0	-1
3	1	0	-1
4	1	1	0
5	1	1	0
6	1	0	-1
⋮			
40			

table of differences

-1	0	1
9	20	11

who went from voting to not voting

Paired t-test calculations

$$\frac{\overline{D} - 0}{\sqrt{s_D/n}}$$

$$\overline{D} = \frac{1}{n}((-1 \times 9) + (0 \times 20) + (1 \times 11)) = \frac{b-c}{n} = \frac{2}{40} = 0.05$$

sample variance of differences

$$\begin{aligned} s_D^2 &= \frac{1}{n-1}\left(9(-1-\overline{D})^2 + 20(0-\overline{D})^2 + 11(1-\overline{D})^2\right) \\ &= \frac{1}{n-1}\left(c + b - \frac{(b-c)^2}{n}\right) \\ &= \frac{1}{40-1}\left(9 + 11 - \frac{(11-9)^2}{n}\right) \\ &= 0.51 \end{aligned}$$

Paired t-test calculations

$$t = \frac{\bar{D}}{\sqrt{\frac{s_D^2}{n}}}$$

no evidence against $p_{before} = p_{after}$

```
##
##  One Sample t-test
##
## data:  df$diff
## t = 0.4427, df = 39, p-value = 0.6604
## alternative hypothesis: true mean is not equal to 0
## 95 percent confidence interval:
##  -0.1784514  0.2784514
## sample estimates:
## mean of x
##      0.05
```

McNemar's test

Null hypothesis: $H_0 : p_{\text{before}} = p_{\text{after}}$

Conditions on the number of discordant pairs, $b + c$.

after

		0	1
before	not vote A **0**	12	11
	vote A **1**	9	8

Under Null hypothesis, we expect the number of discordant pairs (e.g. people who change their minds during debate) should be equally split between b and c.

McNemar's test

Conditional on $b + c$,

$$b \sim \text{Binomial}(b + c, 0.5)$$

Do, one sample Z-test for proportions, leads to

$$Z = \frac{b - c}{\sqrt{b + c}} \dot{\sim} N(0, 1) \quad \text{under null hypothesis}$$

(sometimes people square this statistic, and compare to χ_1^2)

Example: McNemar's

after

		0	1
before	**0**	12	11
	1	9	8

$$Z = \frac{b - c}{\sqrt{b + c}} = \frac{11 - 9}{\sqrt{11 + 9}} = 0.45$$

Compare to N(0,1)

$z_{1-\alpha/2}$

Final points

- McNemar's test is equivalent to the paired t-test, in the sense that the two test statistics are monotone transformations of each other.
- For large sample sizes, the two test statistics get closer and closer to the same value: asymptotically equivalent.