Delta Method and the Bootstrap

ST551 Lecture 27

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Announcements

Lectures this week:

- Today lecture: Delta method and Bootstrap
- Weds lecture: Randomization & Permutation
- Friday lecture: **Cancelled** - Office hours instead

Formula Sheet The final is closed book, no note sheet. I am willing to provide some of the harder (less common) formulae.

Lab: No set material, I'll encourage Chuan to lead a formula strategy session.

Delta Method

Delta Method

If the sampling distribution of a statistic converges to a Normal distribution, the **Delta method**, provides a way to approximate the sampling distribution of a function of a statistic.

Univariate Delta Method

If

$$\sqrt{n}\left(\hat{\theta} - \theta\right) \to_D N(0, \sigma^2)$$

then

$$\sqrt{n}\left(g(\hat{\theta}) - g(\theta)\right) \to_D N(0, \sigma^2[g'(\theta)]^2)$$

(As long as $g'(\theta)$ exists and is non-zero valued.)

Another way of saying it

If we know,

$$\hat{\theta} \mathrel{\dot\sim} N(\theta, \sigma^2)$$

then,

$$g(\hat{\theta}) \mathrel{\dot\sim} N(g(\theta), \sigma^2[g'(\theta)]^2)$$

The approximation can be pretty rough. I.e. just because the sample is large enough that the original statistic is reasonably Normal, doesn't meant the transformed statistic will be.

Example: Log Odds

Let $Y_1, \ldots, Y_n \sim \text{Bernoulli}(p)$, and $X = \sum_{i=1}^{n} Y_i$.

We know $\hat{p} = \frac{X}{n} \sim N(p, \frac{p(1-p)}{n})$.

We might estimate the log odds with:

$$\log\left(\frac{\hat{p}}{1-\hat{p}}\right)$$

What is the assymptotic distribution of the estimated log odds?

Example: Log Odds cont.

$$g(p) = \log\left(\frac{p}{1-p}\right) = \log(p) - \log\left(1-p\right)$$

Other comments on delta method

Derived using a Taylor expansion of $g(\hat{\theta})$ around $g(\theta)$

There is also a multivariate version (useful if you need some function of two statistics, e.g. ratio of sample means)

Bootstrap

Bootstrap

A method to approximate the sampling distribution of a statistic

Idea:

- Recall, one way to approximate the sampling distribution of a statistic was by **simulation**, but you have to assume a population distribution.
- The bootstrap uses the *empirical distribution function* as an estimate for the population distribution, i.e relies on

$$\hat{F}(y) \approx F(y)$$

Example - Sampling distribution of Median by simulation

Assume a population distribution, i.e. $Y \sim N(\mu, \sigma^2)$

Repeat for $k = 1, \ldots, B$

1. Sample n observations from $N(\mu, \sigma^2)$
2. Find sample median, $m^{(k)}$

Then the simulated sample medians, $m^{(k)}, k = 1, \ldots, B$ approximate the sampling distribution of the sample median.

Example - Sampling distribution of Median by bootstrap

Estimate the population distribution from the sample, i.e. $\hat{F}(y)$

Repeat for $k = 1, \ldots, B$

1. Sample n observations from a population with c.d.f $\hat{F}(y)$
2. Find sample median, $m^{(k)}$

Then the bootstrapped sample medians, $m^{(k)}, k = 1, \ldots, B$ approximate the sampling distribution of the sample median.

Sampling from a c.d.f

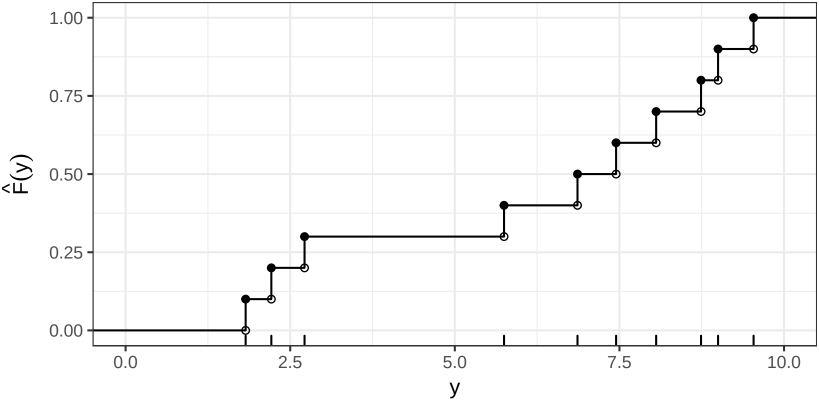
You can sample from any c.d.f by sampling from a Uniform(0, 1), then transforming with the inverse c.d.f.

I.e. sample $u_1, \ldots, u_n$ i.i.d from Uniform(0,1), then

$$y_i = F^{-1}(u_i) \quad i = 1, \ldots, n$$

are distributed with c.d.f $F(y)$.

In the empirical case



Sampling from the ECDF is equivalent to sampling with replacement from the original sample.

Example - Sampling distribution of Median by bootstrap

Repeat for $k = 1, \ldots, B$

1. Sample n observations with replacement from $Y_1, \ldots, Y_n$
2. Find sample median, $m^{(k)}$

Then the bootstrapped sample medians, $m^{(k)}, k = 1, \ldots, B$ approximate the sampling distribution of the sample median.

A little more subtly:

$$\hat{m} - m \mathrel{\dot{\sim}} \tilde{m} - \hat{m}$$

Example

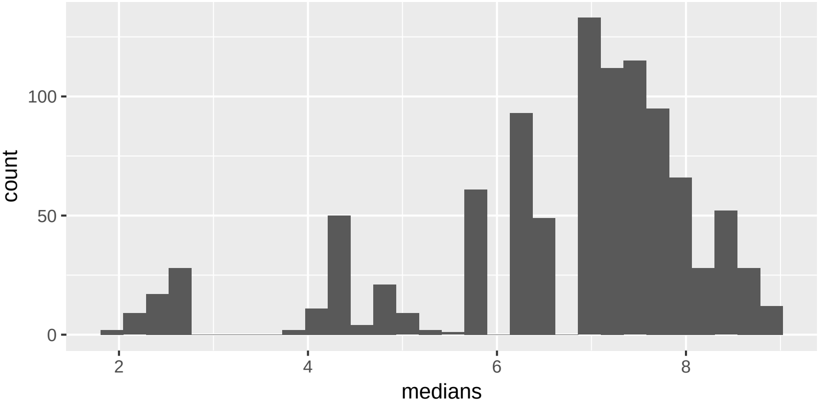
Sample values: *1.8*, *2.2*, *2.7*, *5.7*, *6.9*, *7.4*, *8.1*, *8.7*, *9* and *9.5*

Sample median: 7.1562828

A bootstrap resample: *1.8*, *2.7*, *2.7*, *5.7*, *6.9*, *7.4*, *8.1*, *8.1*, *8.7* and *9.5*

Sample median: 7.1562828

Many resamples



Bootstrap confidence intervals

Many methods..

A common one:

- **Quantile:** $100(\alpha/2)$ largest resampled statistic value, and $100(1-\alpha/2)$ largest resampled statistic value

Comments on the bootstrap

Relies on $\hat{F}(y)$ being a good estimate of the $F(y)$, doesn't necessarily solve small sample problems.

Resampling should generally mimic original study design. E.g. If pairs of observations are sampled from a population, pairs should be resampled