Delta Method and the Bootstrap

ST551 Lecture 2~~6~~7

Charlotte Wickham

2017-11-27

Announcements

Lectures this week:

- Today lecture: Delta method and Bootstrap
- Weds lecture: Randomization & Permutation
- Friday lecture: **Cancelled** - Office hours instead WNGR 255

Formula Sheet The final is closed book, no note sheet. I am willing to provide some of the harder (less common) formulae. Email me suggestions

Lab: No set material, I'll encourage Chuan to lead a formula strategy session.

Delta Method

Delta Method

If the sampling distribution of a statistic converges to a Normal distribution, the **Delta method**, provides a way to approximate the sampling distribution of a function of a statistic. Eg. $\bar{Y} \dot\sim N(\mu, \sigma^2)$

$\bar{Y}^2$

Univariate Delta Method

If

$$\sqrt{n}\left(\underbrace{\hat{\theta}}_{\text{statistic}} - \underbrace{\theta}_{\text{population value}}\right) \rightarrow_D N(0, \overbrace{\sigma^2}^{\text{some variance } Var(\hat{\Theta})})$$

then

derivatives of g at Θ

$$\sqrt{n}\left(\underbrace{g}_{\text{some function}}(\hat{\theta}) - g(\theta)\right) \rightarrow_D N(0, \sigma^2[g'(\theta)]^2)$$

(As long as $g'(\theta)$ exists and is non-zero valued.)

Another way of saying it

If we know,

$$\underbrace{\hat{\theta} \overset{\text{for large samples}}{\dot{\sim}} N(\theta}_{\text{assymptotically unbiased}}, \sigma^2)$$

then,

$$g(\hat{\theta}) \dot{\sim} N(g(\theta), \sigma^2[g'(\theta)]^2)$$

The approximation can be pretty rough. I.e. just because the sample is large enough that the original statistic is reasonably Normal, doesn't meant the transformed statistic will be.

In general $E(g(\hat{\theta})) \neq g(E(\hat{\theta}))$

Example: Log Odds

Let $Y_1, \ldots, Y_n \sim$ Bernoulli(p), and $X = \sum_{i=1}^{n} Y_i$. $= \#$ of 1's

We know $\hat{p} = \frac{X}{n} \overset{\bullet}{\sim} N(p, \underbrace{\frac{p(1-p)}{n}}_{\sigma^2})$.

We might estimate the log odds with:

$$\log\left(\frac{\hat{p}}{1-\hat{p}}\right)$$

What is the assymptotic distribution of the estimated log odds?

Example: Log Odds cont.

$$g(p) = \log\left(\frac{p}{1-p}\right) = \log(p) - \log\left(\underline{1-p}\right)$$

$$g'(p) = \frac{1}{p} + \frac{1}{1-p}$$

(hardest part remember differentiation)

$$= \frac{1}{p(1-p)}$$

$$g(\hat{p}) = \log\left(\frac{\hat{p}}{1-\hat{p}}\right) \dot{\sim} N\left(\log\left(\frac{p}{1-p}\right), \underbrace{\frac{p(1-p)}{n}}_{\sigma^2} \cdot \underbrace{\frac{1}{(p(1-p))^2}}_{(g'(p))^2}\right)$$

$$\dot{\sim} N\left(\log\left(\frac{p}{1-p}\right), \frac{1}{n\, p(1-p)}\right)$$

Other comments on delta method

Derived using a Taylor expansion of $g(\hat{\theta})$ around $g(\theta)$

There is also a multivariate version (useful if you need some function of two statistics, e.g. ratio of sample means)

$$\frac{\bar{X}}{\bar{Y}} \mathrel{\dot\sim}$$

$$\hat{\theta} = (\bar{X}, \bar{Y})$$

$$\theta = (\mu_x, \mu_y)$$

Bootstrap

Bootstrap

A method to approximate the sampling distribution of a statistic

Idea:

- Recall, one way to approximate the sampling distribution of a statistic was by **simulation**, but you have to assume a population distribution.
- The bootstrap uses the *empirical distribution function* as an estimate for the population distribution, i.e relies on

$$\hat{F}(y) \approx F(y)$$

true population c.d.f.

empirical c.d.f based on a sample

Example - Sampling distribution of Median by simulation

Assume a population distribution, i.e. $\underbrace{Y \sim N(\mu, \sigma^2)}$

Repeat for $k = 1, \ldots, B$

1. Sample n observations from $N(\mu, \sigma^2)$
2. Find sample median, $m^{(k)}$

Then the simulated sample medians, $m^{(k)}, k = 1, \ldots, B$ approximate the sampling distribution of the sample median.

Example - Sampling distribution of Median by bootstrap

Estimate the population distribution from the sample, i.e. $\hat{F}(y)$

Repeat for $k = 1, \ldots, B$

1. Sample n observations from a population with c.d.f $\hat{F}(y)$
2. Find sample median, $m^{(k)}$

Then the bootstrapped sample medians, $m^{(k)}, k = 1, \ldots, B$ approximate the sampling distribution of the sample median.

Sampling from a c.d.f

Inverse c.d.f sampling

F(y)

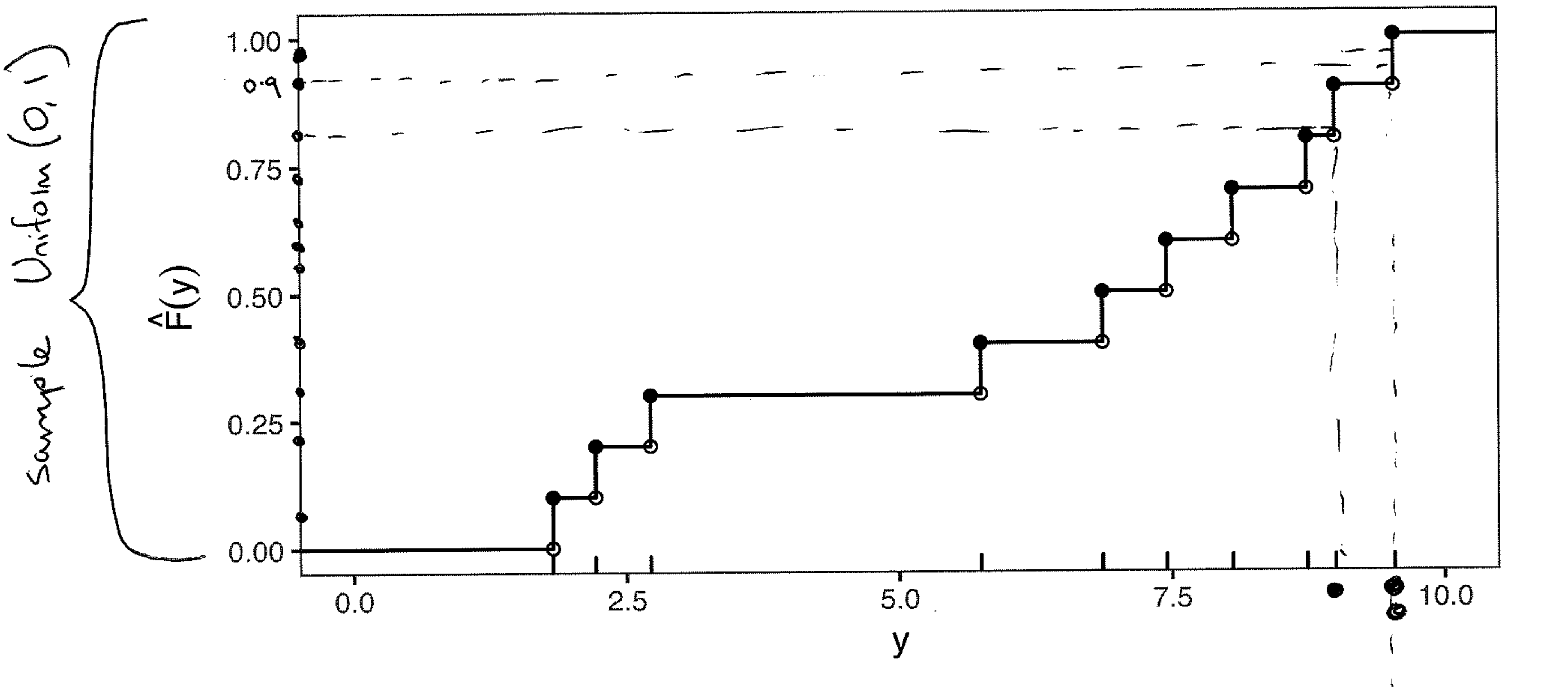
You can sample from any c.d.f by sampling from a Uniform(0, 1), then transforming with the inverse c.d.f.

I.e. sample $u_1, \ldots, u_n$ i.i.d from Uniform(0,1), then

$$y_i = F^{-1}(u_i) \quad i = 1, \ldots, n$$

are distributed with c.d.f $F(y)$.

In the empirical case



Sampling from the ECDF is equivalent to sampling with replacement from the original sample.

Example - Sampling distribution of Median by bootstrap

Repeat for $k = 1, \ldots, B$

1. Sample n observations with replacement from $Y_1, \ldots, Y_n$
2. Find sample median, $m^{(k)}$

Then the bootstrapped sample medians, $m^{(k)}, k = 1, \ldots, B$ approximate the sampling distribution of the sample median.

A little more subtly:

our statistic from the original sample

$$\underbrace{\hat{m} - m} \sim \overbrace{\tilde{m} - \hat{m}}$$

our statistic

true population value

statistic calculated on a sample from the bootstrap population

$$\hat{m} \sim \tilde{m}$$

Example

$\hat{F}(y)$ earlier based on this sample

Sample values: *1.8*, *2.2*, *2.7*, *5.7*, *6.9*, *7.4*, *8.1*, *8.7*, *9* and *9.5*

Sample median: 7.1562828 7.16

A bootstrap resample: *1.8*, *2.7*, *2.7*, *5.7*, *6.9*, *7.4*, *8.1*, *8.1*, *8.7* and *9.5*

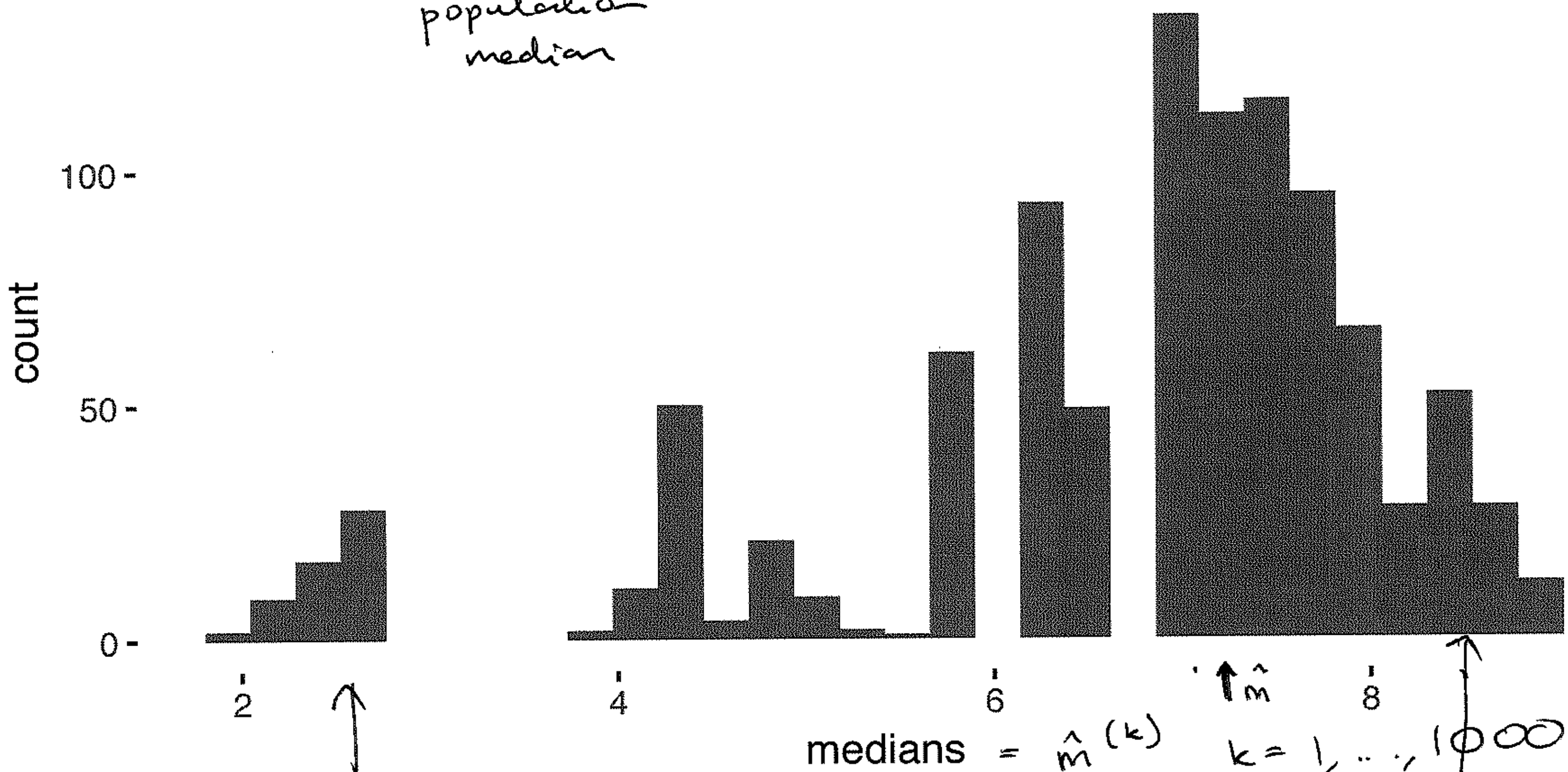
Sample median: 7.156~~2828~~ 7.16 $= \hat{m}^{(1)}$

Many resamples

$$\hat{m} - m \overset{\circ}{\sim} \hat{m}^{(k)} - 7.16$$

population median

$\hat{m}$ based on actual ~~dat~~ data



2.5%ile

97.5th

Bootstrap confidence intervals

Many methods..

A common one:

- **Quantile:** $100(\alpha/2)$ largest resampled statistic value, and $100(1-\alpha/2)$ largest resampled statistic value

Comments on the bootstrap

Relies on $\hat{F}(y)$ being a good estimate of the $F(y)$, doesn't necessarily solve small sample problems.

Resampling should generally mimic original study design. E.g. If pairs of observations are sampled from a population, pairs should be resampled